Ph.D. Qualifying Exam: Algebra
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Determine whether the followings are true or false. If your answer is false, give a counterexample. (You don't have to prove your assertions)
   (a) Let G be a finite group of order n and $d \mid n$. Then there exists a subgroup H of G of order d.
   (b) Any product of free modules is free.
   (c) If L/K is a finite extension of fields, then there exist only finitely many intermediate fields.
   (d) If L is a normal extension of K and K is a normal extension of F, then L is a normal extension of F.

2. [15 pts] Let G be a finite group of order p^3q, where p and q are primes. Then either G has a normal Sylow p-group or Sylow q-group or $p = 2, q = 3$ and $|G| = 24$.

3. [20 pts] Let R be a commutative ring with identity. Show that
   (a) if a is a nilpotent element of R, i.e. $a^m = 0$ for some positive integer m, then $1 + a$ is a unit in R.
   (b) if $a_0 + a_1X + \cdots + a_nX^n$ in $R[X]$ is invertible, then a_0 is a unit and a_i with $i > 0$ is nilpotent.
   (c) if $a_0 + a_1X + \cdots + a_nX^n$ in $R[X]$ is nilpotent, then all a_i's are nilpotent.

4. [15 pts] Let K/F be a Galois extension with cyclic Galois group of order n generated by σ. Suppose that $\alpha \in K$ has $N_{K/F}(\alpha) = 1$. Show that α is of the form $\alpha = \frac{\beta}{\sigma(\beta)}$ for some $\beta \in K$.

5. [15 pts] Prove that the ring
$$\mathbb{Z}[\sqrt{-2}] = \{m + n\sqrt{-2} : m, n \in \mathbb{Z}\}$$
is a Euclidean domain.

6. [15 pts] Let R be a ring with identity. Suppose that
$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\phi'} & C' \end{array}$$
is a commutative diagram of R-modules and that the rows are exact. Prove that
   (a) if ϕ and α is surjective, and β is injective, then γ is injective.
   (b) if ψ', α and γ are injective, then β is injective.

THE END

Ph.D. Qualifying Exam: Differential Geometry
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let $0 < a < b$. Show that the subset of $\mathbb{R}^3$ described by the equation

$$\left(\sqrt{x^2+y^2}-b\right)^2+z^2=a^2$$

is a submanifold. Show that it is diffeomorphic to $S^1 \times S^1$.

2. [20 pts] Suppose that V is a vector space of dimension n and $L^k_{alt}(V)$ is the space of alternating k-multilinear maps.
   (a) Write down the definition of exterior product or wedge product $\alpha \wedge \beta$, with $\alpha \in L^r_{alt}(V)$ and $\beta \in L^s_{alt}(V)$.
   (b) Determine $dim(L^k_{alt}(V))$.
   (c) Suppose $F \in L(V,V)$, that is $F : V \to V$ is a linear map. If $F^* : L^n_{alt}(V) \to L^n_{alt}(V)$ is the map induced on the space of alternating n-multilinear maps, then determine $F^*\omega$ for an arbitrary $\omega \in L^n_{alt}(V)$.

3. [20 pts] Suppose $S \subset M$ is a k-dimensional regular submanifold with boundary ∂S (possibly empty) and Φ_t is the flow of some complete smooth vector field X defined on M.
   (a) Show that $\Phi_t(S)$ is a regular manifold with boundary.
   (b) Show that

$$\int_S \Phi_t^*\eta = \int_{\Phi_t(S)} \eta,$$

   where η is a smooth differential k-form with compact support.
   (c) Show the formula:

$$\frac{d}{dt}\int_{\Phi_t(S)} \eta = \int_{\Phi_t(S)} i_X d\eta + \int_{\partial(\Phi_t(S))} i_X \eta.$$

4. [20 pts] Show the following statement:
   If a smooth n-manifold M has a good cover[1] then its de Rham cohomology groups are all finite dimensional.

5. [20 pts] Compute the de Rham cohomology groups of $\mathbb{R}^n \setminus \{p, q\}$, with p, q distinct points of $\mathbb{R}^n$.

THE END

[1] An open cover $\{U_\alpha\}_{\alpha\in A}$ of an n-manifold is called a good cover if for every choice $\alpha_0, \dots, \alpha_k \in A$ the set $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ is diffeomorphic to $\mathbb{R}^n$ (or empty).

Ph.D. Qualifying Exam: Real Analysis
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Suppose that f is a measurable function in $\mathbb{R}^d$. Prove that there exists a sequence of step functions that converges pointwise to $f(x)$ for almost every x.

2. [15 pts] Suppose that $\mathcal{B} = \{B_1, B_2, \ldots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^d$. Prove that there exists a disjoint subcollection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of $\mathcal{B}$ such that
$$m\left(\bigcup_{\ell=1}^{N} B_\ell\right) \le 3^d \sum_{j=1}^{k} m(B_{i_j}),$$
where m is the Lebesgue measure on $\mathbb{R}^d$.

3. [15 pts] Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying
$$F(x) = \int_a^x f(y)dy$$
for an integrable function f. Prove that F is absolutely continuous.

4. Prove the following statement:
   (a) [10 pts] If $1 \le p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.
   (b) [10 pts] If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

5. [20 pts] Let $\mathcal{H}$ be a Hilbert space and T be a linear bounded operator on $\mathcal{H}$. Prove that the operator norm $\|T\|$ satisfies
$$\|T\| = \sup_{f,g\in\mathcal{H}} \{|\langle Tf, g\rangle : \|f\|, \|g\| \le 1\}.$$

6. [15 pts] Assume that μ, ν, and λ are σ-finite measures on a measure space $(X, \mathcal{M})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Prove that $\nu \ll \lambda$ and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$
almost everywhere.

THE END

Ph.D. Qualifying Exam: Complex Analysis
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let Ω be a connected open subset of $\mathbb{C}$ and $\{f_n\}$ a sequence of injective holomorphic functions on Ω that converges uniformly on every compact subset of Ω to a holomorphic function f. Prove that f is either injective or constant.

2. [10 pts] Compute the following integral and justify the calculation:

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{(x^2+1)^2}\, dx \quad \text{for } k \text{ real.}$$

3. [15 pts] Let $z_1, \cdots, z_n$ be distinct complex numbers contained in the disk $B_R(0) = \{z : |z| < R\}$ for some $R > 0$. And let $q(z) = (z - z_1)\cdots(z - z_n)$. Prove the following:

$$P(z) := \frac{1}{2\pi i} \int_{\partial B_R(0)} \frac{f(\xi)}{\xi - z} \left(1 - \frac{q(z)}{q(\xi)}\right) d\xi$$

is a polynomial of degree $(n-1)$ such that $P(z_k) = f(z_k)$ for $k = 1, \ldots, n$.

4. [15 pts] Let f be a function which is continuous in $\{z \in \mathbb{C} : |z| \leq 1\}$ and holomorphic in $\{z \in \mathbb{C} : |z| < 1\}$. Suppose also that $|f(z)| = 1$ whenever $|z| = 1$. Show that f can be extended to a meromorphic function in $\mathbb{C}$ which has at most a finite number of poles.

5. [15 pts] Let f be a nowhere vanishing holomorphic function in a simply connected open set Ω. Prove that there exists a holomorphic function g in Ω such that $f(z) = e^{g(z)}$.

6. [15 pts] Find all functions, say f, satisfying the following: f is an entire function of finite order that omits two values.

7. [15 pts] Let f be a conformal mapping from $\Omega = \{z : -1 < \mathrm{Re}(z) < 1\}$ to $\{z : |z| < 1\}$ for which $f(0) = 0$ and $f'(0) > 0$. Prove the uniqueness of f and compute $f'(0)$.

THE END

Ph.D. Qualifying Exam: Probability Theory
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let $(\Omega, \mathcal{B}, P)$ be a probability space and $X \in L_1$. For a random variable X', prove the following. $\int_A X\, dP = \int_A X'\, dP$ for any $A \in \mathcal{B}$ if and only if $\int_A X\, dP = \int_A X'\, dP$ for any $A \in \mathcal{P}$ where $\mathcal{P}$ is a π-system generating $\mathcal{B}$ and containing Ω.

2. [15 pts] For a sequence of events $\{A_n\}$, show the following. $P\{\limsup_{n\to\infty} A_n\} = 1$ if and only if $\sum_{n=1}^{\infty} P\{A \cap A_n\} = \infty$ for all events A such that $P\{A\} > 0$.

3. [15 pts] State Central Limit Theorem. Use the following to show Central Limit Theorem.
$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \leq \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right).$$

4. [15 pts] A sequence $\{X_n\}$ is said to converge completely to a random variable X if $\sum_{n=1}^{\infty} P\{|X_n - X| > \epsilon\} < \infty$ for every $\epsilon > 0$. Prove or disprove the following.
   (a) If X_n converges completely to a random variable X, then X_n converges to X almost surely.
   (b) If X_n converges to X almost surely, then X_n converges completely to X.

5. [15 pts] For random variables X_1, X_2 and Y with $Y \in L_1$, suppose that $\sigma(Y, X_1)$ is independent of $\sigma(X_2)$. Show that $E[Y|X_1, X_2] = E[Y|X_1]$ almost surely.

6. [15 pts] If X_n converges in distribution to X_0 and $\sup_{n\geq 1} E[|X_n|^{2+\delta}] < \infty$, then show that $\lim_{n\to\infty} E[X_n] = E[X_0]$ and $\lim_{n\to\infty} Var[X_n] = Var[X_0]$.

7. [10 pts] Show that a sequence of normal distributions is tight if and only if their means and variances are bounded.

THE END

Ph.D. Qualifying Exam: Advanced Statistics
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let $X_1, \cdots, X_n$ be iid from an Exponential distribution with mean $\lambda > 0$. Let $0 < a \leq b < \infty$. For $X_i = x_i$, let

$$Y_i = \begin{cases} 1 & \text{if } 0 < x_i < b \\ 0 & \text{otherwise} \end{cases}$$

and

$$Z_i = \begin{cases} 1 & \text{if } a < x_i \\ 0 & \text{otherwise.} \end{cases}$$

   (a) [5 pts] Find the joint distribution of Y_1 and Z_1.

   (b) [5 pts] Find the condition, if any, on a and b under which Y_1 and Z_1 are independent.

   (c) [10 pts] Suppose that X_i's are not observable and only (Y_i, Z_i), $i = 1, \cdots, n$, are observed. Let $a = b$. Find the MLE of λ based on the observed data.

2. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from the distribution F. Let $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$ where $I(s) = 1$ if the statement s is true and 0 otherwise. Show that $\hat{F}_n(x)$ converges to $F(x)$ in probability.

3. Let $X_1, \cdots, X_n$ be a random sample from $U(\theta_1, \theta_2)$.

   (a) [5 pts] Find the method of moments estimators of θ_1 and θ_2.

   (b) [5 pts] Find the MLE's $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2.

   (c) [10 pts] Find a complete sufficient statistic for (θ_1, θ_2) if it exists. If there is no such statistic, explain why.

   (d) [10 pts] Let $X_{(1)}$ and $X_{(n)}$ be the smallest and the largest of the sample, respectively. Find a 95% confidence interval of $\theta_2 - \theta_1$ and express it in terms of $X_{(1)}$ and $X_{(n)}$.

4. Let $X_1, \cdots, X_n$ be iid from $N(\theta, 1)$. Define

$$Y_i = \begin{cases} 1 & \text{if } X_i > 0 \\ 0 & \text{if } X_i \leq 0. \end{cases}$$

Let $\psi = P(Y_i = 1)$.

(a) [3 pts] Find the MLE $\hat{\theta}$ of θ.

(b) [3 pts] Find the MLE $\hat{\psi}$ of ψ.

(c) [4 pts] Let $\tilde{\psi} = (\sum_{i=1}^n Y_i)/n$. Find $Var(\tilde{\psi})$.

(d) [10 pts] Find an asymptotic variance of $\hat{\psi}$ by applying the Delta method.

5. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from a distribution with parameter θ. Let $\hat{\theta}$ be the MLE of θ. Show that, for any function g, the MLE $\hat{\tau}$ of $\tau = g(\theta)$ is given by $g(\hat{\theta})$. This is the invariance property of the MLE.

6. [10 pts] Let $X_1, \cdots, X_n$ be a random sample from a Poisson distribution with parameter λ. Let the observed value of $\sum_{i=1}^n X_i = y_0$. Use the equation

$$P(W \leq w) = P(Z \geq k)$$

where W is a Gamma(k, β) random variable and Z a Poisson(w/β) random variable, and find a 95% confidence interval of λ.

THE END

Ph.D. Qualifying Exam: Numerical Analysis
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts]
   (a) Define a Lagrange interpolation polynomial with data $\{(x_i, f(x_i))\}_{i=0}^{n}$ for x_i all distinct.
   (b) What is the error form in the above? Derive it.
   (c) Define a Newton's form of interpolation polynomial using the same data.
   (d) Explain what happens if some x_i are repeated in the Newton's form. What is the correct data corresponding to the repeated points?

2. [15 pts] Describe Newton's method to solve a system of nonlinear equations $\mathbf{F}(\mathbf{x}) := A(\mathbf{x})\mathbf{x} + \mathbf{g}(\mathbf{x}) + \mathbf{b} = 0$ starting from some initial point $\mathbf{x}_0$. Here $\mathbf{x} = (x, y)$ and $\mathbf{b} = (b_1, b_2)$ are nonzero vectors, $A = (a_{ij}(\mathbf{x}))$ is a 2×2 nonsingular matrix of variable entries $a_{ij}(\mathbf{x})$, and $\mathbf{g}(\mathbf{x}) \in \mathbb{R}^2$ is a C^1 vector function of $\mathbf{x}$. In describing, include a specific form of the derivative of $\mathbf{F}(\mathbf{x})$.

3. [10 pts] Explain Runge phenomena in approximation theory and suggest how one can avoid it.

4. [20 pts] Define the following basic numerical methods to find the roots of a real valued equation $f(x) = 0$ when $x \in \mathbb{R}$. Discuss advantage/disadvantage, condition for convergence, convergence rate, etc.
   (a) Bisection method
   (b) Picard's method
   (c) Newtons method
   (d) Secant method

5. [15 pts] Prove the following Schur lemma: If $M \in \mathbb{C}^{n,n}$, then $\exists$ a unitary matrix U such that $U^H M U = T$, where T is upper triangular.

   Hint: Start with an eigenpair $(\lambda_1, \mathbf{u})$ of M with $\|\mathbf{u}\|^2 = 1$, find an orthogonal matrix U_1 such that
   $$U_1 \mathbf{e}_1 = \mathbf{u}.$$

6. [20 pts] Explain briefly the following methods for computing eigenvalues of certain matrices. Discuss applicability, effectiveness, etc.
   (a) Jacobi algorithm
   (b) Givens algorithm
   (c) QR iterations
   (d) Power method

THE END

Ph.D. Qualifying Exam: Combinatorics
February 2016

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20pts] Let n be a positive integer. Let G be the complete graph on $\binom{n}{3}$ vertices such that $V(G)$ is the set of all 3-element subsets of $\{1, 2, \ldots, n\}$. We color each edge $\{A, B\}$ by red if $|A \cap B| = 1$ and by blue otherwise.

   Prove that $R(n+2, n+1) > \binom{n}{3}$ by using this coloring of G.

   (Here, $R(m, n)$ denotes the minimum N such that every coloring of the edges of K_N into two colors red and blue induces a red K_m subgraph or a blue K_n subgraph.)

2. [20pts] Prove that every graph G contains a 3-colorable subgraph H such that $|E(H)| \geq \frac{2}{3}|E(G)|$.

3. [20pts] Let $f(n)$ be the maximum number of edges in an n-vertex simple bipartite graph without a cycle of length 4. Prove that $f(n) \leq (1 + o(1))n^{3/2}$.

4. [20pts] Let $k \geq 2$. Suppose that $\mathcal{F}$ is a set of subsets of $\{1, 2, \ldots, n\}$. Prove that if $\mathcal{F}$ contains no k members $A_1, A_2, \ldots, A_k$ such that $A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_k$, then
$$\sum_{X \in \mathcal{F}} \frac{1}{\binom{n}{|X|}} \leq k - 1.$$

5. [20pts] Let $M = (m_{ij})_{i,j}$ be a symmetric $n \times n$ matrix satisfying the following.

   (i) $m_{i,j} \in \{1, 2, \ldots, n\}$ for all $i, j \in \{1, 2, \ldots, n\}$.

   (ii) $m_{i,j} \neq i$ and $m_{i,j} \neq j$ for all $i \neq j$.

   Prove that there exists a subset X of $\{1, 2, \ldots, n\}$ such that $|X| \geq \frac{1}{2}\sqrt{n}$ and $m_{i,j} \notin X$ for all $i, j \in X$ with $i \neq j$.

   Hint: Take a random subset Y and do something to get X from Y.

THE END